Intro GGT Exercises 1

Exercise 1. Let K be a field. Consider the action of $\mathrm{GL}_n(K)$ on the vector space K^n of column vectors.

1. Show that this action has exactly two orbits.
2. Determine the stabilizer of the vector
$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$
3. Show that this stabilizer surjects onto $\mathrm{GL}_{n-1}(K)$, and that the kernel of this surjection is isomorphic to K^{n-1}.
4. Deduce that if K is a finite field with q elements, then
$$|\mathrm{GL}_n(K)| = (q^n - 1)q^{n-1}|\mathrm{GL}_{n-1}(K)|.$$
5. Compute explicitly the cardinality of $\mathrm{GL}_n(K)$ when K is a finite field.

Exercise 2 (Cube)**.** Consider a cube, for instance the convex hull in $\mathbb{R}^3$ of the points $(\pm 1, \pm 1, \pm 1)$. Let G be the subgroup of $O(3)$ preserving this cube.

1. Given two adjacent vertices A, B, show that there exists an element of G sending A to B. Deduce that G acts transitively on the vertices and that $|G|$ is divisible by 8.
2. Given two adjacent edges U, V, show that there exists an element of G sending U to V. Deduce that G acts transitively on edges, and even on oriented edges. Deduce that $|G|$ is a multiple of 24.
3. Show that the stabilizer of an oriented edge is the reflection across the plane containing it, and deduce that $|G| = 48$.
4. Show that the center of G is $Z(G) = \{\pm \mathrm{id}\}$, and that G decomposes as
$$G = G^+ \times \{\pm \mathrm{Id}\}.$$
5. Show that $G^+ = G \cap \mathrm{SO}(3)$ has 24 elements.
6. Consider the 4 long diagonals of the cube. Show that G^+ acts transitively on this set, and deduce an isomorphism $G^+ \simeq S_4$.

Exercise 3 (Hausdorff distance)**.** Let (X, d) be a metric space. For a nonempty subset $A \subset X$, define
$$d(x, A) := \inf_{a \in A} d(x, a).$$
Denote by $\mathcal{B}(X)$ the set of *nonempty closed bounded subsets* of X. For $A, B \in \mathcal{B}(X)$, define the Hausdorff distance
$$d_H(A, B) := \max \left\{ \sup_{a \in A} d(a, B), \ \sup_{b \in B} d(b, A) \right\}.$$
We will prove that $(\mathcal{B}(X), d_H)$ is a metric space.

1. Show that if A is bounded, then $\sup_{a \in A} d(a, B)$ is finite.

2. Give an example of two subsets $A, B \subset \mathbb{R}$, possibly unbounded, such that $d_H(A, B)$ is infinite. Explain why the assumption of boundedness is necessary.

3. Show that for all $x \in X$ and all subsets $A \subset X$, $d(x, A) = 0$ if and only if $x \in \overline{A}$.

4. Prove that for $A, B \in \mathcal{B}(X)$, $d_H(A, B) = 0$ in and only if $A = B$.

5. Give an example of two bounded subsets $A, B \subset \mathbb{R}$, possibly not closed, such that

$$d_H(A, B) = 0 \quad \text{but } A \neq B.$$

Explain why the assumption of closedness is necessary.

6. Show that for all nonempty subsets $A, B, C \subset X$ and all $a \in A$,

$$d(a, C) \leq d(a, B) + \sup_{b \in B} d(b, C).$$

Hint: Fix $\varepsilon > 0$ and choose $b \in B$ and $c \in C$ such that

$$d(a, b) \leq d(a, B) + \varepsilon \quad \text{and} \quad d(b, c) \leq d(b, C) + \varepsilon.$$

7. Show that

$$\sup_{a \in A} d(a, C) \leq \sup_{a \in A} d(a, B) + \sup_{b \in B} d(b, C).$$

8. Conclude that for all nonempty closed bounded subsets $A, B, C \subset X$,

$$d_H(A, C) \leq d_H(A, B) + d_H(B, C).$$

9. Conclude that $(\mathcal{B}(X), d_H)$ is a metric space.

Exercise 4 (Cayley graphs and direct products)**.** Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be graphs. The *Cartesian product* $\Gamma_1 \times \Gamma_2$ is the graph with $V_1 \times V_2$ as the set of vertices, and $\{(v_1, v_2), (v_1', v_2')\}$ is an edge if and only if either

- $v_1 = v_1'$ and $\{v_2, v_2'\} \in E_2$, or
- $v_2 = v_2'$ and $\{v_1, v_1'\} \in E_1$.

1. Show that $\Gamma_1 \times \Gamma_2$ is connected if and only if both Γ_1 and Γ_2 are connected.

2. Show that the graph distance in $\Gamma_1 \times \Gamma_2$ satisfies

$$d_{\Gamma_1 \times \Gamma_2}\big((v_1, v_2), (v_1', v_2')\big) = d_{\Gamma_1}(v_1, v_1') + d_{\Gamma_2}(v_2, v_2').$$

3. Let G, H be groups with finite generating sets S_G and S_H respectively. Show that $S := (S_G \times \{e_H\}) \cup (\{e_G\} \times S_H)$ is a generating set of $G \times H$.

4. Construct an explicit graph isomorphism

$$\text{Cay}(G \times H, S) \longrightarrow \text{Cay}(G, S_G) \times \text{Cay}(H, S_H).$$

5. Deduce that for all $(g, h), (g', h') \in G \times H$,

$$d_S\big((g, h), (g', h')\big) = d_{S_G}(g, g') + d_{S_H}(h, h').$$

6. Apply this to draw a Cayley graph of $\mathbb{Z} \times D_n$, with a generating set of your choice.

7. Apply this to draw a Cayley graph of $\mathbb{Z}/3\mathbb{Z} \times Z/3\mathbb{Z}$, with a generating set of your choice.